15 November 1996

ELSEVIER

Optics Communications 132 (1996) 35–40

OPTICS COMMUNICATIONS

# Phase quantization effects on Fresnel lenses encoded in low resolution devices

E. Carcolé [a], J. Campos [b], I. Juvells [c]

[a] *Universitat Politècnica de Catalunya, Escola Universitària d'Òptica i Optometria de Terrassa, Departament d'Òptica i Optometria, Violinista Vellsolà 37, 08222 Terrassa, Spain*

[b] *Universitat Autònoma de Barcelona, Departament de Física, 08193 Bellaterra, Spain*

[c] *Universitat de Barcelona, Departament de Física Aplicada i Electrònica, Laboratori d'Òptica, Diagonal 647, 08028 Barcelona, Spain*

Received 22 December 1995; revised version received 5 March 1996; accepted 24 April 1996

**Abstract**

The low resolution and phase quantization involved in the codification of a Fresnel lens in a spatial light modulator produce non-negligible effects in the performance quality. In this paper we study the effects due to both limitations in such devices.

*Keywords:* Diffraction; Fresnel lenses; Zone plates; Spatial light modulators; Digital holography

## 1. Introduction

The Fresnel lens is one of the simplest and most useful holograms which can be encoded in a spatial light modulator (SLM). The SLM permits changing holograms at frame speed. This implies the possibility of using Fresnel lenses encoded in SLM as variable focal length lenses in optical setups [1,2]. Also, a wide variety of applications of holograms including low resolution Fresnel encoded lenses (LRFELs) have been studied [3–7]. Mostly, SLMs are low resolution devices. Then, for a Fresnel lens, this involves the low resolution encoding of a quadratic phase. This fact implies non-negligible effects in the Fresnel lens performance quality. Recently, Carcolé et al. have developed a theory to describe the performance of the LRFEL [8]. This theory is based on a mathematical model of the LRFEL that makes it possible to use the diffraction theory in the Fresnel approximation [9]. Using this theory the amplitude distribution in the focal plane is calculated for all focal regions. The resulting expressions are functions of several adimensional parameters that take into account the characteristics of the SLM, the wavelength and the focal length encoded. By using this theory it is also possible to derive expressions for the diffraction efficiency corresponding to each focal region [10]. A way to optimize short focal length lenses has also been developed [11].

Another basic limitation of the SLM is that only a certain number of phase levels can be encoded. The purpose of this paper is double: first, to complete the description of low resolution effects dealt with in Ref. [8] and second, to use the new results to describe the effects of the phase quantization of the LRFEL. In Ref. [12] a general description of phase quantization effects on phase functions is developed and we are going to use it.

The theoretical background of this paper is contained in Refs. [8,12]; then, in Section 2, the main results of these two papers are explained in relation


*PII* S0030-4018(96)00321-5

to our development. In Section 3 we complete the description of low resolution effects on Fresnel lenses in order to enable the development in Section 4. In Section 4 we describe the effects of phase quantization on the LRFEL using the results of Section 3. In Section 5 the particular case of binary Fresnel lenses is considered and the effect of phase quantization and low resolution on the shape of the lens itself are shown. In Section 6 the conclusions of the paper are presented.

## 2. Theoretical background

The purpose of this section is to introduce the basic notation and results that we shall use throughout the paper and are taken from Ref. [8] (in Section 2.1) and Ref. [12] (Section 2.2).

### 2.1. Sampling effects encoding Fresnel lenses

When a Fresnel lens with focal length $f$ is sampled with an infinite matrix of Dirac delta functions with period $\Delta x$ and $\Delta y$ in the $x$ and $y$ direction respectively, new focalizations appear in the focal plane at the coordinates $(kX, lY)$ where $k$, $l$ are integer numbers and $X$ and $Y$ are defined by

$$X = \frac{\lambda f}{\Delta x}, \quad Y = \frac{\lambda f}{\Delta y}. \tag{1}$$

Then an LRFEL seems to be a matrix of lenses. The apparent size of each lens is $XY$. The number of lenses that lies inside a rectangular device with dimensions $L_x$, $L_y$ is given by

$$W_x = \frac{L_x}{X}, \quad W_y = \frac{L_y}{Y}. \tag{2}$$

In Fig. 1 we can see a single LRFEL for $W_x = W_y = 7$.

Another interesting property is that, keeping the sampling matrix in the same position, sampling a Fresnel lens centered at the (0, 0) coordinate is completely equivalent to sampling another Fresnel lens centered at the $(kX, lY)$ coordinate ($k$, $l$ being arbitrary integer numbers) with a phase shift given by

$$\begin{aligned}\phi_{(k,l)} = & -\pi\left[k^2 R_x + l^2 R_y\right] \\ & -2\pi\left[k\left(\tfrac{1}{2}P(N) - D(kR_x)\right)\right. \\ & \left.+ l\left(\tfrac{1}{2}P(M) - D(lR_y)\right)\right],\end{aligned} \tag{3}$$

where

$$R_x = \frac{X}{\Delta x}, \qquad R_y = \frac{Y}{\Delta y}, \tag{4}$$

and $N$ and $M$ are the number of delta functions in the $x$ and $y$ direction that lie inside a rectangle with dimensions $L_x$, $L_y$ and $P(x)$ is a function that was used in order to take into account whether the lens is centered at a delta function (then $x$ is an odd natural number and $P(x) = 0$) or between two delta functions (then $x$ is an even natural number and $P = 1$); $D(x)$ means the fractional part of $x$. Note that, in the general case, we can consider the center of the lens to be the coordinate $(a\Delta x/2, b\Delta y/2)$ where $a$ and $b$ are arbitrary real numbers. This is equivalent to considering $P(N) = a$ and $P(M) = b$.

Finally, an important conclusion of Ref. [8] is that the complex amplitude originated by the propagation of any distribution encoded in a low resolution device (for instance a Fresnel lens) can be calculated: first we calculate the propagation supposing that the pixels are perfect points (mathematically, delta Dirac functions), and finally we convolve the amplitude obtained in this way with the function that defines the transmittance of a single pixel. So, through the paper, the functions representing a transmittance function will not be considered to be convolved by the transmittance function of a single pixel and will be written as a linear combination of delta functions.

### 2.2. Phase quantization effects

In Ref. [12] it is shown that, when a phase function $\exp(i\varphi)$ is quantized in $L$ levels, the new resulting function $\exp(i\varphi_L)$ can be written as

$$\begin{aligned}\exp(i\varphi_L) = & \sum_{m=-\infty}^{\infty} \operatorname{sinc}(1/L + m) \\ & \times \exp\left[i(Lm + 1)\varphi\right],\end{aligned} \tag{5}$$

where $m$ takes integer values. If $\varphi$ corresponds to the phase of a Fresnel lens with focal length $f$, i.e.

$$\varphi = -\frac{K}{2f}(x^2 + y^2) \tag{6}$$

($K = 2\pi/\lambda$ where $\lambda$ is the wavelength) then, from Eq. (5), the Fresnel lens becomes the sum of Fresnel lenses with different focal lengths (each one identified by a different value of $m$) given by

$$f_m = f/(Lm + 1). \tag{7}$$

For $L = 2$ we get the known result for Fresnel zone plates:

$$f_m = f/(2m+1). \tag{8}$$

## 3. Effects of low resolution on a Fresnel lens

In order to sample a Fresnel lens for encoding in a low resolution device we use the following function:

$$\mathrm{comb}(x, y) \equiv \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\{x - [n + \tfrac{1}{2}P(N)]\Delta x, \quad y - [m + \tfrac{1}{2}P(M)]\Delta y\}, \tag{9}$$

where $\delta(x, y)$ is the two dimensional delta Dirac function, $n$ and $m$ are integers, and we multiply it by the lens transmission,

$$z_f(x, y) = \exp\left[-\mathrm{i}\frac{K}{2f}(x^2 + y^2)\right], \tag{10}$$

resulting in

$$z_{f,\mathrm{d}}(x, y) = z_f(x, y)\mathrm{comb}(x, y), \tag{11}$$

where $z_{f,\mathrm{d}}(x, y)$ is the sampled Fresnel lens. comb$(x, y)$ is a periodic function, so it can be expanded into a two-dimensional Fourier series:

$$z_{f,\mathrm{d}}(x, y) = \frac{1}{\Delta x\,\Delta y} z_f(x, y) \times \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \exp\left[2\pi\mathrm{i}\left(\frac{k}{\Delta x}x + \frac{l}{\Delta y}y\right)\right] \times \exp[-\mathrm{i}\pi(kP(N) + lP(M))], \tag{12}$$

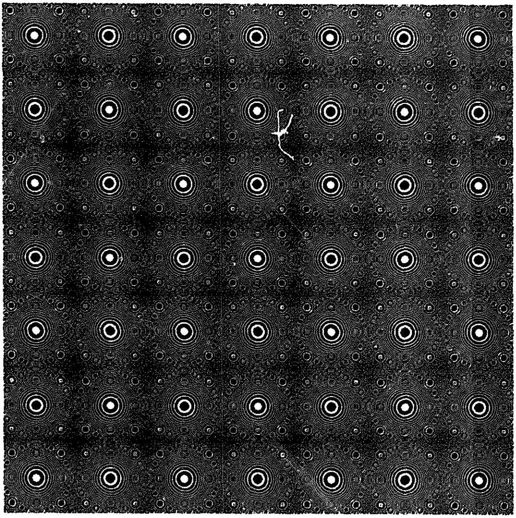

Fig. 1. LRFEL corresponding to $W_x = W_y = 7$.

where $k$ and $l$ are integer numbers. Using Eq. (12) in Eq. (11) it is easy to get

$$z_{f,d}(x, y) = \frac{1}{\Delta x\, \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \exp\left(-i\frac{K}{2f}\left[(x-kX)^2 + (y-lY)^2\right]\right) \exp\left[i\frac{k}{2f}\left[(kX)^2+(lY)^2\right]\right] \times \exp\left[-i\pi(kP(N)+lP(M))\right]. \tag{13}$$

This equality can be rewritten in the following way:

$$z_{f,d}(x, y) = \frac{1}{\Delta x\, \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} z_f \times (x-kX, y-lY) \exp(i\phi_{(k,l)}), \tag{14}$$

where

$$\phi_{(k,l)} = \pi\left[k^2R_x + l^2R_y\right] - \pi\left[kP(N) + lP(M)\right]. \tag{15}$$

Eq. (14) means that a sampled Fresnel lens can also be understood as an infinite sum of shifted Fresnel lenses, each one identified by the pair of numbers $(k, l)$. The shifts are multiples of the $(X, Y)$ quantities, and each lens has a phase shift given by $\phi_{(k,l)}$ in Eq. (15). Note that when we are encoding a real LRFEL, for example the one shown in Fig. 1, a pupil function always affects all the new shifted Fresnel lenses. Note also that the actual size of this pupil for each $(k, l)$ lens is not $XY$ (the apparent size of each shifted Fresnel lens in Fig. 1) but $L_x L_y$.

Then, note that if we encode a matrix of 49 identical Fresnel lenses using copies of a Fresnel lens of size $XY$, we will obtain something that will look like Fig. 1. But the point spread function that will be obtained in the focal plane will be much wider because the actual size of the pupil function of each lens of the matrix is $XY$. Fig. 1 comes from the sampling of a single Fresnel lens, and is actually an infinite set of multiplexed Fresnel lenses, and the size of each lens is $L_x = 7X$ and $L_y = 7Y$ while the point spread function will be narrower.

From Eq. (14) it is easy to get

$$z_{f,d}(x, y) = z_{f,d}(x-kX, y-lY) \exp(i\phi_{k,l}),$$

where $k$ and $l$ are arbitrary integer numbers. This means that it is the same to encode a sampled Fresnel lens centered at the (0, 0) coordinate as it is to encode another Fresnel lens centered at the $(kX, lY)$ coordinate with a phase shift $\phi_{k,l}$. This was first derived in Ref. [8] in a rather cumbersome way and commented in Section 2. Then we expect that the phase $\phi_{k,l}$ given by Eqs. (15) and (3) are the same phase. To make Eqs. (3) and (15) equal, it is necessary to add to Eq. (3) the following number: $2\pi[kE(kR_x) + lE(lR_y)]$, where $E(x)$ means the integer part of $x$, and to take into account that $D(x) + E(x) = x$. Note that the number to add is just a multiple of $2\pi$ so we are not changing the actual value of the phase. Eqs. (14) and (15) are the final result that completes our description of the low resolution encoding effects of a Fresnel lens. This result enables us to develop the following section.

## 4. Phase quantization effects on the LRFEL

In this section, we shall analyze the effects of encoding an LRFEL in a device that only permits a certain number of phase levels to be encoded. For this purpose we use the trivial fact that first sampling and then quantizing the phase is completely equivalent to first quantizing the phase and then sampling it. For a quantized Fresnel lens with focal length $f$, $z_{f,L}(x, y)$ can be written, using Eqs. (5) and (6), as

$$z_{f,L} = \sum_{m=-\infty}^{\infty} \mathrm{sinc}(1/L + m)\; z_{f_m}(x, y). \tag{16}$$

This means that a phase quantized Fresnel lens with focal length $f$ is equivalent to the sum of infinite Fresnel lenses with focal length $f_m$ given in Eq. (7). If we sample it, i.e. multiply it by the comb function as in Eqs. (11) and (12), note that this is the same as sampling each $z_{f_m}(x, y)$ in Eq. (16), so a phase quantized LRFEL $z_{f,d,L}(x, y)$ can be written as

$$z_{f,d,L}(x, y) = \sum_{m=-\infty}^{\infty} \mathrm{sinc}(1/L + m)\; z_{f_m,d}(x, y). \tag{17}$$

Now, using Eqs. (14) and (15), we get

$$z_{f,\mathrm{d},L}(x,y)=\frac{1}{\Delta x\,\Delta y}\sum_{m=-\infty}^{\infty}\mathrm{sinc}(1/L+m)\times\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}z_{f_m}(x-kX_m,\,y-lY_m)\times\exp(\mathrm{i}\phi_{(k,l,m)}),\tag{18}$$

with

$$\phi_{(k,l,m)}=\pi\left(k^2R_{m,x}+l^2R_{m,y}\right)-\pi(kP(N)+lP(M))\tag{19}$$

and

$$X_m=\frac{\lambda f_m}{\Delta x},\quad Y_m=\frac{\lambda f_m}{\Delta y},\quad R_{m,x}=\frac{X_m}{\Delta x},\quad R_{m,y}=\frac{Y_m}{\Delta y}.\tag{20}$$

The meaning of Eqs. (18), (19) and (20) is the following. Each $z_{f_m}(x, y)$ coming from phase quantization becomes, due to the sampling, the sum of infinite new Fresnel lenses, identified by the $(k, l)$ numbers, and with the same focal length $f_m$. Each $(k, l)$ lens is shifted by a distance $(kX_m, lY_m)$ from the origin and has a phase shift given by $\phi_{(k,l,m)}$. This implies that in each focal plane defined by $f_m$ we get an infinite matrix of focalizations with positions given by $(kX_m, lY_m)$ with $k$, $l$ arbitrary integer numbers and with a phase shift given by $\phi_{(k,l,m)}$. The amplitude of all focalizations in an $f_m$ plane is affected by the factor $\mathrm{sinc}(1/L+m)$. Eqs. (18), (19) and (20) are the key result of this paper and they completely describe a phase quantized LRFEL. Because of its intrinsic importance and in order to illustrate these results we shall study in detail the case $N=2$ (binary optics).

## 5. Binary LRFEL

Let us consider a binary LRFEL with $W_x=W_y=1$ and focal length $f$. A lens satisfying this equality is shown in Fig. 2(a). The binarization of the LRFEL implies the existence of infinite focal planes. If we illuminate it with a plane wave the position of these planes is given by $f_m=f/(2m+1)$. In each focal plane (identified by a value of $m$) there is no single focalization as in a regular zone plate; due to the low resolution effects we always have infinite focalizations due to each $(k, l)$ lens in Eq. (18). At distance $f$ we have a matrix of focalizations with coordinates $(kX, lY)$. At the distance $f/3$, we now also have a matrix of focalizations with coordinates $(kX/3, lY/3)$ as given by Eqs. (20) and (8), with $L=2$ and $m=1$.

We shall also show, graphically rather than mathematically, that the combination of phase quantiza-

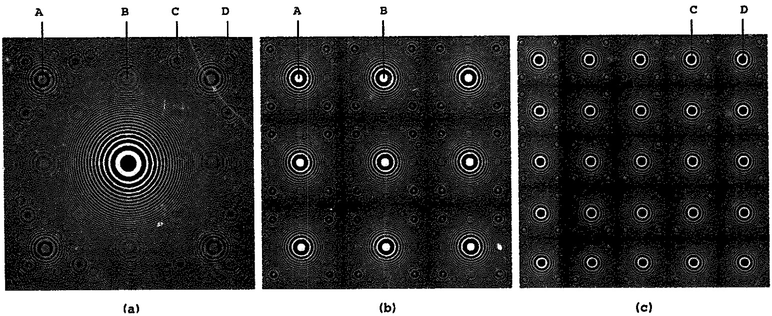


Fig. 2. (a) Binary LRFEL corresponding to $W_x=W_y=1$. (b) Binary representation of $-(z_{f/3,\mathrm{d}}+z_{-f/3,\mathrm{d}})$. (c) Binary representation of $(z_{f/5,\mathrm{d}}+z_{-f/5,\mathrm{d}})$. Capital letters label lenses from Figs. 2(b) and 2(c) that can be identified in Fig. 2(a).

tion and low resolution effects can be noticed in the shape of an LRFEL, as for example the one in Fig. 2(a). In this way we will justify the visual aspect of the binary LRFEL. At the distances $f/3$ and $-f/3$ the focalizations corresponding to $-1 \leqslant k \leqslant 1$ and $-1 \leqslant l \leqslant 1$ lie inside the geometric projection of the pupil of our lens. The lens that causes all the focalizations at $f/3$ is $-z_{f/3,\mathrm{d}}$ and the one that causes focalizations at $-f/3$ is $-z_{-f/3,\mathrm{d}}$ (these are the terms $m = 1$ and $m = -2$ in Eq. (17), the negative sign affecting the lenses coming from $\mathrm{sinc}(1 + \frac{1}{2})$ and $\mathrm{sinc}(-2 + \frac{1}{2})$, and it has been taken into account that $f_1 = f/3$ and $f_{-2} = -f/3$). We will now show, that it can be noticed in the shape of the LRFEL in Fig. 2(a) that it contains $-(z_{f/3,\mathrm{d}} + z_{-f/3,\mathrm{d}})$.

The representation of $-(z_{f/3,\mathrm{d}} + z_{-f/3,\mathrm{d}})$ is drawn in Fig. 2(b) and for simplicity we have drawn it binary. This representation is equivalent to the representation of the binary LRFEL corresponding to $-z_{f/3,\mathrm{d}}$. After careful examination, note that each $(k, l)$ lens of Fig. 2(b) can now be identified in Fig. 2(a). To make it easier, two lenses are labeled as A and B. Then the effects of phase quantization in each focal plane can also be noticed in the lens itself. We can do the same for $f/5$ and $-f/5$. This corresponds to Fig. 2(c) and we can also identify the lenses corresponding to $-2 \leqslant k \leqslant 2$ and $-2 \leqslant l \leqslant 2$ in Fig. 2(a). Two of them are labeled as C and D. It is interesting to note that it is more difficult to see the lenses corresponding to $f/5$ and $-f/5$ than the ones corresponding to $f/3$ and $-f/3$. Certainly, the 'visibility' of the lenses from Figs. 2(b), (c) in Fig. 2(a) probably depends on the value of $\mathrm{sinc}(m + 1/N)$. Although we do not prove it, in any binary LRFEL it always seems possible to notice the combination of binarization and low resolution effects in the same way as we did for $W_x = W_y = 1$.

## 6. Conclusions

In this paper we have shown the following:

(i) A sampled Fresnel lens is equivalent to the sum of infinite Fresnel lenses with the same focal length. Each lens resulting from the sampling process is identified by the pair of numbers $(k, l)$ and the coordinates of the center of each lens are given by $(kX, lY)$. Each lens has a phase shift if compared with the $(0, 0)$ lens. We have found the expression for this phase in terms of the parameters that define the LRFEL.

(ii) A phase quantized Fresnel lens can be written as a sum of infinite on-axis Fresnel lenses with different focal length. This implies the existence of infinite focal planes. The sampling of phase quantized Fresnel lenses is equivalent to the sum of sampled on-axis lenses resulting from the phase quantization. From conclusion (i), each one can be written as a new infinite set of shifted Fresnel lenses and each one has a phase shift given by our theory.

(iii) For the specific case of binary LRFEL, we have shown that the effects of phase quantization can be noted in the pattern of the lens itself. In a binary LRFEL the lenses corresponding to $f/3$ and $-f/3$, $f/5$ and $-f/5, \ldots$, and its phases can be seen in the lens itself. In this way we explain the visual aspect of the binary Fresnel lens.

## Acknowledgements

This work was supported in part by the Comisión Interministerial de Ciencia y Tecnología, Spain, projects TAP93-0667-C03-02, TAP93-0667-C03-01, ROB91-0554. E.C. acknowledges a Formació d'Investigadors grant from the Generalitat de Catalunya.

## References

[1] E.C. Tam, S. Zhou and M.R. Feldman, Appl. Optics 31 (1992) 578.
[2] E. Carcolé, J. Davis and D. Cottrell, Appl. Optics 34 (1995) 5118.
[3] Don M. Cottrell, J.A. Davis, T.R. Hedman and R.A. Lilly, Appl. Optics 29 (1990) 2505.
[4] J.A. Davis, W.V. Brandt, D.M. Cottrell and R.M. Bunch, Appl. Optics 30 (1991) 4610.
[5] J.A. Davis and D.M. Cottrell, Appl. Optics 19 (1994) 496.
[6] J.A. Davis, D.M. Cottrell, J.E. Davis and R.A. Lilly, Appl. Optics 14 (1989) 659.
[7] J.A. Davis, S.H. Drayton, D.M. Cottrell and J.E. Davis, Appl. Optics 29 (1990) 2594.
[8] E. Carcolé, J. Campos and S. Bosch, Appl. Optics 33 (1994) 162.
[9] J.W. Goodman, Introduction a l'Optique de Fourier et a l'Holographie (Masson, Paris, 1972) pp. 53–72.
[10] E. Carcolé, J. Campos, Y. Juvells and S. Bosch, Appl. Optics 33 (1994) 6741.
[11] E. Carcolé, J. Campos, I. Juvells and J. Moneo, Appl. Optics 34 (1995) 5952.
[12] W.J. Dallas, Appl. Optics 10 (1971) 673.